Apunte 40

Escalón unitario

$$f(t) = \begin{cases} \sin t & 0 \leq t < \frac{\pi}{4} \\ \sin(t) + \cos(t) & t \geq \frac{\pi}{4} \end{cases}$$

Escribir en términos de la función escalón unitario.

$$f(t) = \sin t + g(t) \text{ con } g(t) = \begin{cases} 0 & 0 \leq t < \frac{\pi}{4} \\ \cos(t) & t \geq \frac{\pi}{4} \end{cases}$$

$$g(t) = u\left(t - \frac{\pi}{4}\right)\cos(t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{g(t)\}$$

$$= \frac{1}{s^2+1} + \mathcal{L}\left\{u\left(t - \frac{\pi}{4}\right)\cos(t)\right\}$$

$$= \frac{1}{s^2+1} + e^{\frac{-\pi}{4}s}\frac{s}{s^2+1} = \frac{1 + se^{-\frac{\pi}{4}s}}{s^2+1}$$

$$u\left(t - \frac{\pi}{4}\right) = \begin{cases} 0 & 0 \leq t < \frac{\pi}{4} \\ 1 & t \geq \frac{\pi}{4} \end{cases}$$

Resolver:

$$y' + y = f(t) \qquad y(0) = 5 \qquad f(t) = \begin{cases} 0 & 0 \le t < \pi \\ 3\cos t & t \ge \pi \end{cases}$$

$f(t) = u(t-\pi)3\cos(t)$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$sY(s) - y(0) + Y(s) = \frac{3s}{s^2+1}e^{-\pi s}$$

$$Y(s)(s+1) - 5 = \frac{3s}{s^2+1}e^{-\pi s}$$

$$Y(s) = \frac{\left(\frac{3s}{s^2+1}e^{-\pi s}\right)}{(s+1)} + \frac{5}{(s+1)}$$

$$Y(s) = \frac{5}{s+1} + \frac{3e^{-\pi s}s}{(s^2+1)(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}s}{(s^2+1)(s+1)}\right\}$$

$$= 5e^{-t} + 3\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}s}{(s^2+1)(s+1)}\right\}$$

Obteniendo a:

$$\left\{\frac{s}{(s^2+1)(s+1)}\right\}$$

$$\frac{s}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\begin{aligned} s &= A(s^2+1) + (Bs+C)(s+1) \\ &= As^2 + A + Bs^2 + Bs + Cs + C \\ &= (A+B)s^2 + (B+C)s + (A+C) \end{aligned}$$

El sistema:

$A + C = 0 \rightarrow A = -C$

$$\left.\begin{matrix} B + C = 1 \\ A + B = 0 \end{matrix}\right\} \begin{matrix} -A + B = 1 \\ \underbrace{A + B = 0}_{2B=1} \end{matrix} \rightarrow \begin{matrix} B = \frac{1}{2} \quad C = \frac{1}{2} \\ A = -\frac{1}{2} \end{matrix}$$

$$\frac{s}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} = \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{2}s}{s^2+1} + \frac{\frac{1}{2}}{s^2+1}$$

Obteniendo así:

$$= 5e^{-t} + \frac{3}{2}\mathcal{L}^{-1}\left\{-\frac{e^{-\pi s}}{s+1} + \frac{e^{-\pi s}s}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}\right\}$$

$\mathcal{L}\{f(t-a)u(t-a)\}$
$= e^{-as}F(s), F(s) = \mathcal{L}\{f\}$

$$= \frac{3}{2}\left[-e^{-(t-\pi)}u(t-\pi) + \cos(t-\pi)\,u(t-\pi) + \sin(t-\pi)\,u(t-\pi)\right]$$

La solución general de la ED es:

$$\therefore y(t) = 5e^{-t} + \frac{3}{2}\left[-e^{-(t-\pi)}u(t-\pi) + \cos(t-\pi)\,u(t-\pi) + \sin(t-\pi)\,u(t-\pi)\right]$$

$$= 5e^{-t} + u(t-\pi)\left[\frac{3}{2}\cos(t-\pi) + \frac{3}{2}\sin(t-\pi) - \frac{3}{2}e^{-t+\pi}\right]$$

Resolver:

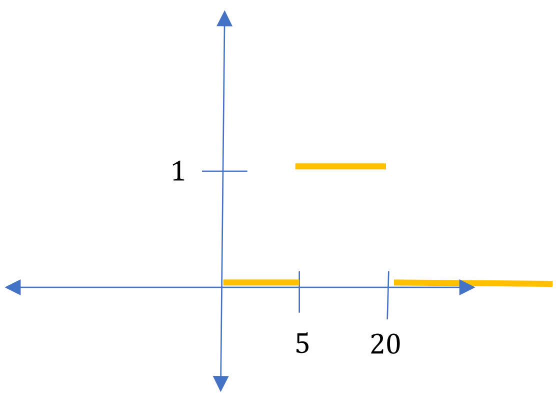
$$2y'' + y' + 2y = g(t) \quad , \quad y(0) = 0 \ , \ y'(0) = 0$$

$$g(t) = u(t-5) - u(t-20) = \begin{cases} 1 & 5 \le t \le 20 \\ 0 & 0 \le t \le 5 \quad y \quad t \ge 20 \end{cases}$$

Graficar a g.

$$u(t-5) = \begin{cases} 0 & 0 \le t < 5 \\ 1 & t \ge 5 \end{cases} \quad , \quad u(t-20) = \begin{cases} 0 & 0 \le t < 20 \\ 1 & t \ge 20 \end{cases}$$

Grafica de $g(t)$:



$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = \mathcal{L}\{g(t)\}$$

$$Y(s)(2s^2 + s + 2) = \mathcal{L}\{u(t-5)\} - \mathcal{L}\{u(t-20)\}$$

$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$
Donde $F(s) = \mathcal{L}\{f(t)\}$

$\mathcal{L}\{u(t-a) * 1\} = e^{-as} * \frac{1}{s}$

$F(s) = \mathcal{L}\{1\} = \frac{1}{s}$

$$Y(s)(2s^2 + s + 2) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$Y(s) = \frac{e^{-5s}}{s(2s^2 + s + 2)} - \frac{e^{-20s}}{s(2s^2 + s + 2)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}\right\}$$

$$\frac{1}{s(2s^2 + s + 2)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2}$$

$$1 = A(2s^2 + s + 2) + (Bs + C)s$$
$$1 = 2As^2 + As + 2A + Bs^2 + Cs$$
$$1 = (2A + B)s^2 + (A + C)s + 2A$$

$$A = \frac{1}{2} \quad , \quad A + C = 0 \rightarrow C = -\frac{1}{2} \quad , \quad 2A + B = 0 \rightarrow B = -2A = -1$$

$$\frac{1}{s(2s^2 + s + 2)} = \frac{\frac{1}{2}}{s} + \frac{\left(-s - \frac{1}{2}\right)}{2s^2 + s + 2}$$

$$\frac{\frac{1}{2}}{s} + \frac{\left(-s - \frac{1}{2}\right)}{2s^2 + s + 2} = \frac{\frac{1}{2}}{s} - \frac{\left(s + \frac{1}{2}\right)}{2\left(s^2 + \frac{s}{2} + 1\right)}$$

$$\left(s + \frac{1}{2}\right) = \left(s + \frac{1}{4}\right) + \frac{1}{4}$$

$$s^2 + \frac{s}{2} + 1 = s^2 + \frac{s}{2} + \frac{1}{16} - \frac{1}{16} + 1$$
$$= \left(s + \frac{1}{4}\right)^2 + \frac{15}{16}$$

$$= \frac{\frac{1}{2}}{s} - \frac{1}{2}\frac{\left(s + \frac{1}{4}\right) + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}$$

$$= \frac{\frac{1}{2}}{s} - \frac{1}{2}\left[\frac{\left(s + \frac{1}{4}\right)}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}} + \frac{\sqrt{15}\,\frac{1}{4}}{\frac{1}{\sqrt{15}}\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}\right] = F(s)$$

$$y(t) = [e^{-5s} - e^{-20s}]F(s) \qquad , \qquad F(s) = \frac{1}{s(2s^2 + s + 2)}$$

$$= e^{-5s}F(s) - e^{-20s}F(s)$$

Aplicando transformada inversa:

$$= \mathcal{L}^{-1}\{e^{-5s}F(s)\} - \mathcal{L}^{-1}\{e^{-20s}F(s)\}$$

$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$

Por lo tanto, la solución general de la ED es la expresión:

$$= u(t-5)f(t-5) - u(t-20)f(t-20)$$

Donde $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2} - \frac{1}{2}\left[\cos\left(\frac{\sqrt{15}}{4}t\right)e^{-\frac{t}{4}} + \frac{\sin\left(\frac{\sqrt{15}}{4}t\right)e^{-\frac{t}{4}}}{\sqrt{15}}\right]$$

Evaluando a $f(t)$ para $(t-5)$:

$$= u(t-5)\left(\frac{1}{2} - \frac{1}{2}e^{-\frac{t-5}{4}}\cos\left(\frac{\sqrt{15}(t-5)}{4}\right) - \frac{e^{-\frac{t-5}{4}}\sin\left(\frac{\sqrt{15}(t-5)}{4}\right)}{2\sqrt{15}}\right)$$

Evaluando $f(t)$ para $(t-20)$:

$$= u(t-20)\left(\frac{1}{2} - \frac{1}{2}e^{-\frac{t-20}{4}}\cos\left(\frac{\sqrt{15}(t-20)}{4}\right) - \frac{e^{-\frac{t-20}{4}}\sin\left(\frac{\sqrt{15}(t-20)}{4}\right)}{2\sqrt{15}}\right)$$