

Apunte 39

Transformada de Laplace

Teorema: Si $f, f', \dots, f^{(n-1)}$ son continuas en $[0, +\infty)$ y son de orden exponencial y si $f^{(n)}(t)$ es continua por tramos en $[0, +\infty)$, entonces:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - s^0 f^{n-1}(0)$$

Donde $F(s) = \mathcal{L}\{f(t)\}$

Si:

$$n = 1$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$n = 2$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$n = 3$$

$$\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

Resolver:

$$y' + 3y = 13 \sin(2t) \quad , \quad y(0) = 6$$

i. Aplicar \mathcal{L}

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin(2t)\}$$

$$sY(s) - y(0) + 3Y(s) = 13 * \frac{2}{s^2 + 2^2}$$

Donde $Y(s) = \mathcal{L}\{y\}$

$$Y(s)(s + 3) - 6 = \frac{26}{s^2 + 4}$$

$$Y(s) = \frac{\left(\frac{26}{s^2 + 4} + 6\right)}{s + 3} = \frac{\frac{26}{s^2 + 4}}{\frac{s + 3}{1}} + \frac{6}{s + 3}$$

$$Y(s) = \frac{26}{(s^2 + 4)(s + 3)} + \frac{6}{s + 3}$$

ii. Aplicar y^{-1} $Y(s) = \mathcal{L}\{y\}$ entonces $y = \mathcal{L}^{-1}\{Y(s)\}$

$$y = \mathcal{L}^{-1}\left\{\frac{26}{(s^2 + 4)(s + 3)}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s + 3}\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{26}{(s^2 + 4)(s + 3)}\right\} + 6e^{-3t}$$

Obteniendo a:

$$\mathcal{L}^{-1}\left\{\frac{26}{(s^2 + 4)(s + 3)}\right\}$$

$$\frac{26}{(s^2 + 4)(s + 3)} = \frac{As + B}{s^2 + 4} + \frac{C}{s + 3}$$

$$26 = (As + B)(s + 3) + C(s^2 + 4)$$

$$\begin{aligned} &= As^2 + 3As + Bs + 3B + Cs^2 + 4C \\ &= (A + C)s^2 + (3A + B)s + (3B + 4C) \end{aligned}$$

Sistema:

$$A + C = 0 \rightarrow A = -C$$

$$\left. \begin{array}{l} 3A + B = 0 \\ 3B + 4C = 26 \end{array} \right\} \rightarrow \begin{array}{l} -3C + B = 0 \quad (4) \\ 4C + 3B = 26 \quad (3) \end{array} \quad \left| \begin{array}{l} -12C + 4B = 0 \\ 12C + 9B = 78 \end{array} \right. \\ \hline 13B = 78$$

$$\rightarrow B = \frac{78}{13} = 6$$

$$3A = -B$$

$$A = \frac{-B}{3} = \frac{-6}{3} = -2, \quad C = 2$$

Por lo tanto:

$$\mathcal{L}^{-1}\left\{\frac{26}{(s^2 + 4)(s + 3)}\right\} = \frac{As + B}{s^2 + 4} + \frac{C}{s + 3} + 6e^{-3t}$$

$$\rightarrow = \mathcal{L}^{-1}\left\{\frac{-2s + 6}{s^2 + 4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s + 3}\right\} + 6e^{-3t}$$

$$= -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 6e^{-3t}$$

Obteniendo transformadas inversas:

$$\begin{aligned} &= -2(\cos 2t) + 6\left(\frac{1}{2}\sin 2t\right) + 2(e^{-3t}) + 6e^{-3t} \\ &= -2\cos(2t) + 3\sin(2t) + 8e^{-3t} \end{aligned}$$

Por lo que la solución general es:

$$y(t) = -2\cos(2t) + 3\sin(2t) + 8e^{-3t}$$

Aplicando las condiciones iniciales:

$$y(0) = 6$$

$$6 = y(t) = -2\cos(0) + 3\sin(0) + 8e^0$$

$$6 = -2 + 0 + 8$$

$$6 = 6$$

Por lo tanto, se cumple el PVI.

Resolver:

$$y'' - 3y' + 2y = e^{-4t} \quad y(0) = 1, \quad y'(0) = 5$$

i. Aplicar \mathcal{L}

$$\mathcal{L}^{-1}\{y''\} - 3\mathcal{L}^{-1}\{y'\} + 2\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

Donde $Y(s) = \mathcal{L}\{y\}$

Sustituyendo:

$$s^2Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{1}{s+4}$$

$$s^2Y(s) - s - 5 - 3sY(s) + 3 + 2Y(s) = \frac{1}{s+4}$$

Factorizando:

$$Y(s)(s^2 - 3s + 2) - s - 2 = \frac{1}{s+4}$$

Despejando:

$$Y(s) = \frac{\left(\frac{1}{s+4} + s + 2\right)}{(s^2 - 3s + 2)}$$

$$Y(s) = \frac{\frac{1}{s+4}}{\frac{1}{s^2 - 3s + 2}} + \frac{s + 2}{s^2 - 3s + 2}$$

$$Y(s) = \frac{1}{(s+4)(s^2 - 3s + 2)} + \frac{s + 2}{s^2 - 3s + 2}$$

ii. Aplicar y^{-1} $Y(s) = \mathcal{L}\{y\}$ entonces $y = \mathcal{L}^{-1}\{Y(s)\}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{(s+3)^2}{(s+4)(s-1)(s-2)}\right\}$$

Obteniendo a:

$$\mathcal{L}^{-1}\left\{\frac{(s+3)^2}{(s+4)(s-1)(s-2)}\right\}$$

$$\frac{(s+3)^2}{(s-2)(s+4)(s-1)} = \frac{A}{s-2} + \frac{B}{s+4} + \frac{C}{s-1}$$

$$(s+3)^2 = A(s+4)(s-1) + B(s-2)(s-1) + C(s-2)(s+4)$$

$$\begin{array}{l} s = 1 \quad 16 = C(-1)(5) \\ s = 2 \quad 25 = A(6)(1) \\ s = -4 \quad 1 = B(-6)(-5) \end{array} \quad \begin{array}{l} C = -\frac{16}{5} \\ A = \frac{25}{6} \\ B = \frac{1}{30} \end{array}$$

Por lo tanto:

$$\begin{aligned} y(t) &= \frac{A}{s-2} + \frac{B}{s+4} + \frac{C}{s-1} = \frac{25}{6} \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30} \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - \frac{16}{5} \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t} - \frac{16}{5} e^t \end{aligned}$$

Por lo tanto, la solución general es:

$$y(t) = \frac{25}{6} e^{2t} - \frac{16}{5} e^t + \frac{1}{30} e^{-4t}$$

Aplicando las condiciones iniciales:

$$y(0) = 1 \quad , \quad y'(0) = 5$$

$$1 = y(0) = \frac{25}{6}e^0 - \frac{16}{5}e^0 + \frac{1}{30}e^0$$

$$1 = y(0) = 1$$

$$5 = y'(0) = \frac{25}{6}(2)e^0 - \frac{16}{5}e^0 + \frac{1}{30}(-4)e^0$$

$$= \frac{25}{3} - \frac{16}{5} - \frac{2}{15} = 5$$

$$5 = y'(0) = 5$$

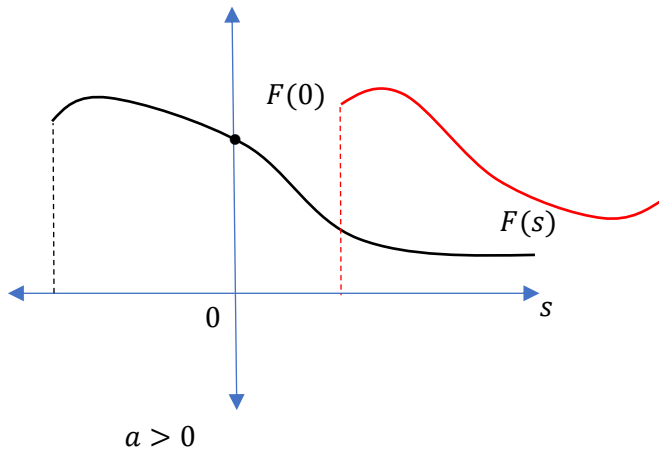
Por lo tanto, se cumple el PVI.

Teorema: Si $\mathcal{L}\{f(t)\} = F(s)$ y $a \in \mathbb{R}$, entonces:

$$\rightarrow \mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}$$

$s \rightarrow s - a$



Transformada de:

(1)

$$\mathcal{L}\{e^{4t}t^3\} = \left\{ \frac{3!}{s^4} \right\}_{s \rightarrow s-4} = \frac{3!}{(s-4)^4}$$

$$a = 4, \quad f(t) = t^3 \quad \mathcal{L}\{t^3\} = \frac{3!}{s^4}$$

(2)

$$\mathcal{L}\{e^{3t} \sin(6t)\} = \mathcal{L}\{\sin(6t)\}_{s \rightarrow s-3} =$$

$$a = 3 \quad f(t) = \sin(6t)$$

$$= \frac{6}{s^2 + 36} \Big|_{s \rightarrow s-3}$$

$$= \frac{6}{(s-3)^2 + 36}$$

(3)

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s \rightarrow s-a} = F(s-a)$$

Donde $F(s) = \mathcal{L}\{f(t)\}$

Entonces:

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$$

(4)

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = y^{-1}\left\{\frac{2s}{(s-3)^2} + \frac{5}{(s-3)^2}\right\}$$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$\left. \begin{aligned} 2s+5 &= A(s-3) + B \\ &= As + (-3A+B) \end{aligned} \right\} \begin{aligned} A &= 2 \\ -3A+B &= 5 \end{aligned} \quad , \quad B = 11$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = 2e^{3t} + 11te^{3t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} \quad , \quad F(s) = \frac{1}{s^2} \quad F(s-3) = \frac{1}{(s-3)^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = t$$

$$\rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = e^{3t}t$$

Resolver:

$$y'' - 6y' + 9y = t^2 e^{3t} \quad , \quad y(0) = 2 \quad , \quad y'(0) = 17$$

i. Aplicar \mathcal{L}

$$\mathcal{L}^{-1}\{y''\} - 6\mathcal{L}^{-1}\{y'\} + 9\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\{t^2 e^{3t}\}$$

$$s^2 Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2!}{s^3} \Big|_{s=3} = \frac{2!}{(s-3)^2}$$
$$f(t) = t^2 \quad a = 3$$

Donde $Y(s) = \mathcal{L}\{y\}$

Sustituyendo:

$$s^2 Y(s) - sy(0) - y'(0) - 6sY(s) + 6y(0) + 9Y(s) = \frac{2!}{(s-3)^3}$$

$$s^2 Y(s) - 2s - 17 - 6sY(s) + 12 + 9Y(s) = \frac{2!}{(s-3)^3}$$

Factorizando:

$$Y(s)(s^2 - 6s + 9) - 2s - 5 = \frac{2!}{(s-3)^3}$$

Despejando:

$$Y(s) = \frac{2!}{(s-3)^3(s^2 - 6s + 9)} + \frac{2s + 5}{(s^2 - 6s + 9)}$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2s + 5}{(s-3)^2}$$

ii. Aplicar y^{-1} $Y(s) = \mathcal{L}\{y\}$ entonces $y = \mathcal{L}^{-1}\{Y(s)\}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\} + \mathcal{L}\left\{\frac{2s+5}{(s-3)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\} + 2e^{3t} + 11te^{3t}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\}$$

$$2\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^5}\right\} \quad F(s) = \frac{1}{s^5} \quad F(s-3) = \frac{1}{(s-3)^5}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}t^4 = \frac{t^4}{4!} = \frac{1}{12}t^4 \quad , \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^5}\right\} = \frac{e^{3t}t^4}{4!} = \frac{1}{12}e^{3t}t^4$$

Por lo tanto, la solución general es:

$$y(t) = \frac{1}{12}e^{3t}t^4 + 2e^{3t} + 11te^{3t}$$

Aplicando las condiciones iniciales:

$$y(0) = 2 \quad , \quad y'(0) = 17$$

$$2 = y(t) = \frac{1}{12}e^{0}0^4 + 2e^0 + 11(0)e^0$$

$$2 = y(t) = 2$$

$$17 = y'(0) = \frac{1}{12}(3)e^{3t}t^4 + \frac{1}{12}(4)e^{3t}t^{4-1} + 2(3)e^{3t} + 11(3)te^{3t} + 11e^{3t}$$

$$= \frac{1}{4}e^{3t}t^4 + \frac{1}{3}e^{3t}t^3 + 33te^{3t} + 17e^{3t}$$

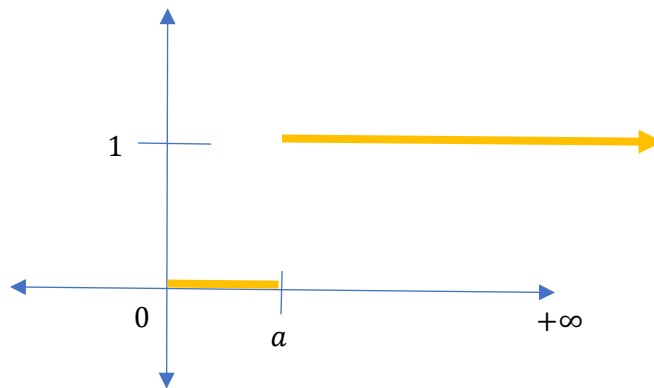
$$= \frac{1}{4}e^{0}0^4 + \frac{1}{3}e^{0}0^3 + 33(0)e^0 + 17e^0$$

$$17 = y(t) = 17$$

Definición: Función escalón unitario.

Esta función, $u(t - a)$, se define como:

$$u(t - a) = \begin{cases} 0 & 0 \leq t \leq a \\ 1 & t \geq a \end{cases}$$



$$f(t) = \begin{cases} g(t) & 0 \leq t \leq a \\ h(t) & t > a \end{cases}$$

(Ejercicio)