Apunte 38

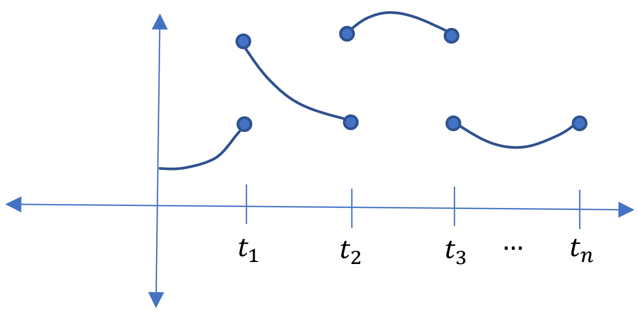
Transformada de Laplace.

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\, dt$$

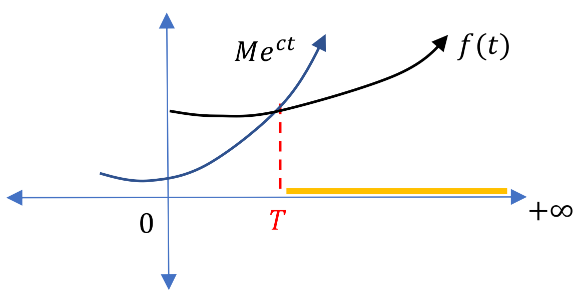
Condiciones suficientes que garantizan la existencia de $\mathcal{L}\{f(t)\}$

i) Sea f continua por tramos $[0, +\infty)$

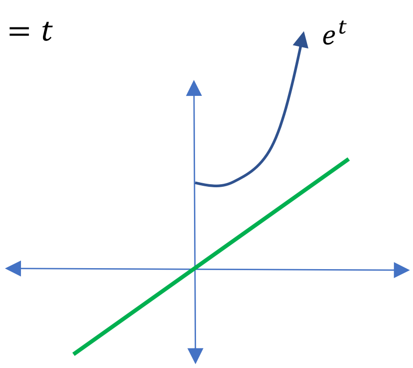
Si existe un número finito de puntos $t_k \quad k = 1, \dots, n \quad (t_{k-1} < t_k)$ en los que f es discontinua y además el $\lim_{t \to t_k^+} f(t) \quad y \quad \lim_{t \to t_k^-} f(t) \quad existen.$



ii) f es de orden exponencial c si existen $c, M > 0$ y $T > 0 \quad \ni \quad |f(t)| \leq Me^{ct} \quad \forall t > T.$



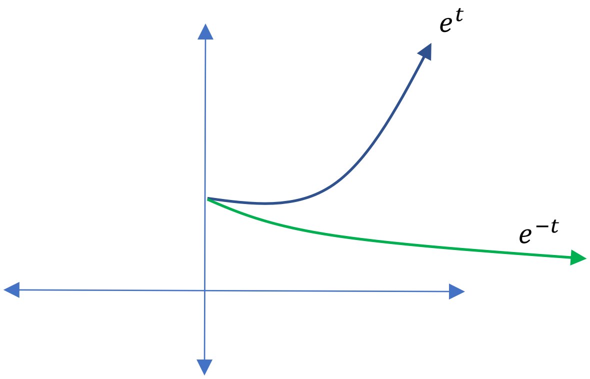
$f_1(t) = t$



$|t| \leq Me^{ct} = e^t \quad M = 1 \quad c = 1$

f_1 es de orden exponencial $c = 1$

$f_2(t) = e^{-t}$

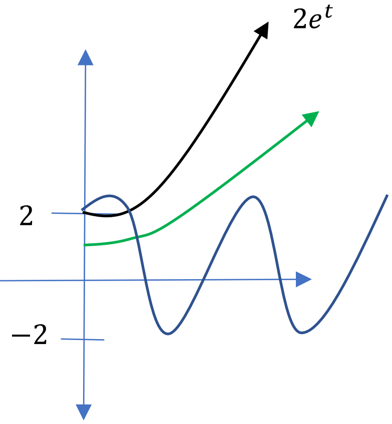


$|f_2(t)| \leq Me^{ct} = e^t$

$M = 1 \quad c = 1$

f_2 es de orden exponencial $c = 1$

$f_3(t) = 2\cos t$



$|2\cos t| \leq Me^{ct}$

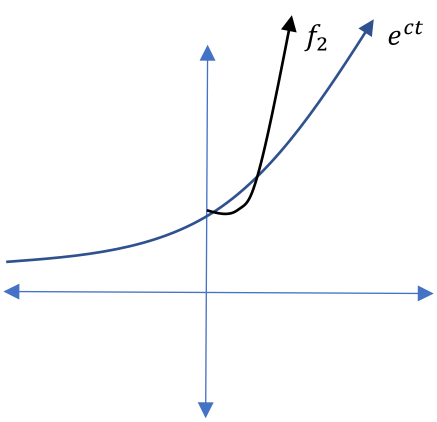
$M = 2 \quad c = 1$

$M = 1 = c$

$M = 2 \quad c = 1$

f_3 es de orden exponencial $c = 1$

$f_4(t) = e^{t^2}$

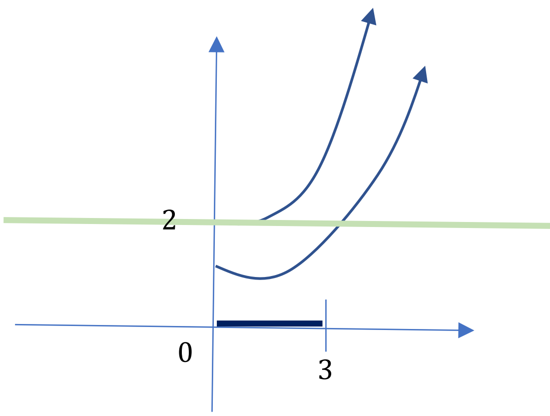


f_4 no es de orden exponencial dado que e^{t^2}, su gráfica crece más rápido que cualquier potencia lineal positiva de e.

Teorema: Si f es una continua por tramos en $[0, +\infty)$ y de orden exponencial c, entonces: $\mathcal{L}\{f(t)\}$ existe para $s > c$

$$f(t) = \begin{cases} 0 & 0 \leq t \leq 3 \\ 2 & t \geq 3 \end{cases}$$

¿ Existe $\mathcal{L}\{f(t)\}$?



i) f si es continua por tramos.

ii) $M = 1 \quad c = 1$

$M = 2 \quad c = 1$

$\rightarrow M = 2 \quad c = 0$

$$|f(t)| \leq 2e^{0t} = 2$$

Existe $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\, dt = \int_0^3 e^{-st} * 0\, dt + \int_3^{\infty} e^{-st} * 2$$

$$= \lim_{A\to\infty} \int_3^A e^{-st} * 2\, dt = \lim_{A\to\infty} 2\left[\frac{-e^{-sA}}{s} + \frac{e^{-3s}}{s}\right] = \underbrace{\frac{2e^{-3s}}{s}}_{F(s)} \quad si \quad s > 0 = c$$

$$s > 0$$
$$A > 0$$

$$s < 0$$
$$A > 0$$

$$\lim_{A\to\infty} \frac{-e^{-sA}}{s} = -\infty$$

Teorema: Si f es continua por partes en $(0,+\infty)$ y de orden exponencial y $F(s)=\mathcal{L}\{f(t)\}$ entonces:

$$\lim_{s\to\infty} F(s) = 0$$

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3} \underset{s\to\infty}{\longrightarrow} 0$$

$$\mathcal{L}\{t^2\} = \frac{s}{s^2+4^2} \underset{s\to\infty}{\longrightarrow} 0$$

$$f(t) \underset{\mathcal{L}}{\to} F(s)$$

$$f(t) \underset{\mathcal{L}^{-1}}{\longrightarrow} F(s)$$

Si $F(s)=\mathcal{L}\{f(t)\}$ entonces $f(t)=\mathcal{L}^{-1}\{F(s)\}$

Por ejemplo:

Transformadas	Transformadas inversas
$\mathcal{L}\{1\} = \frac{1}{s}$	$1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$
$\mathcal{L}\{t\} = \frac{1}{s^2}$	$t = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$
$\mathcal{L}\{e^{-4t}\} = \frac{1}{s+4}$	$e^{-4t} = \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$

Ejemplo 1: Encontrar transformada inversa

$$F(s) = \frac{1}{s^5} \qquad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad n + 1 = 5 \rightarrow n = 4$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{4!}t^4 = f(t)$$

Ejemplo 2: Encontrar transformada inversa

$$F(s) = \frac{1}{s^2 + 11} \qquad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s + a} \qquad \mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}$$

$$b^2 = 11 \rightarrow b = \sqrt{11}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 11}\right\} = \frac{1}{\sqrt{11}}\mathcal{L}^{-1}\left\{\frac{\sqrt{11}}{s^2 + 11}\right\} = \frac{1}{\sqrt{11}}\sin\sqrt{11}\,t$$

$\mathcal{L}^{-1}$ es una transformada lineal.

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$$

Ejemplo 3: Encontrar transformada inversa

$$F(s) = \frac{-2s + 6}{s^2 + 4} \quad , \quad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{-2s + 6}{s^2 + 4}\right\} = -2\mathcal{L}^{-1}\left\{\frac{5}{s^2 + 4}\right\} + \frac{6}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}$$

$$= -2\cos(2t) + 3\cos(2t)$$

Ejemplo 4: Encontrar transformada inversa

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

$$s^2+6s+9=A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)$$

$$s=1 \qquad 16=A(-1)(5)\rightarrow A=-\frac{16}{5}$$

$$s=2 \qquad 25=B(-1)(6)\rightarrow B=-\frac{25}{6}$$

$$s=-4 \qquad 1=C(-5)(-6)\rightarrow C=\frac{1}{30}$$

$$=\frac{-\frac{16}{5}}{s-1}-\frac{\frac{25}{6}}{s-2}+\frac{\frac{1}{30}}{s+4}$$

$$=-\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-\frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$=-\frac{16}{5}e^{t}-\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}$$

Teorema: Si $f, f', \dots, f^{(n-1)}$ son continuas en $[0, +\infty)$ y son de orden exponencial y si $f^{(n)}(t)$ es continua por tramos en $[0, +\infty)$, entonces:

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s^0 f^{n-1}(0)$$

Donde $F(s) = \mathcal{L}\{f(t)\}$

Si:

$n = 1$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$n = 2$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$n = 3$

$$\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$