Apunte 36

Resolver:

$$2xy'' + (1+x)y' + y = 0$$

Punto singular $x_0 = 0$ ¿Regular? Si

Por el Teorema de Frobenius: Existe una solución:

$$y = \sum_{n=0}^{\infty} C_n x^{n+r}$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)C_n x^{n+r}$$

$$+\sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)C_n x^{n+r} = 0$$

$$= r(2r-1)C_0 x^{r-1} + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)C_n x^{n+r} = 0$$

Factorizando a x^r

$$x^r\left(r(2r-1)C_0 x^{-1} + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)C_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)C_n x^n\right) = 0$$

Cambio de variable:

$$m = n-1 \quad , \quad m = n$$

$$n = m+1$$

$$x^r\left(r(2r-1)C_0 x^{-1} + \sum_{m=0}^{\infty}(m+1+r)(2m+2r+1)C_{m+1}x^m + \sum_{m=0}^{\infty}(m+r+1)C_m x^m\right) = 0$$

Haciendo la conversión a una sola suma:

$$x^r\left(r(2r-1)C_0x^{-1}+\sum_{m=0}^{\infty}[(m+1+r)(2m+2r+1)C_{m+1}+(m+r+1)C_m]\,x^m\right)=0$$

Dado que x^r, donde x no puede ser 0, ya que es un punto singular donde estamos calculando la serie. Por lo que los coeficientes deben ser cero.

Sean los coeficientes:

$$r(2r-1)=0$$

$$(m+1+r)(2m+2r+1)C_{m+1}+(m+r+1)C_m=0$$

Por lo tanto,

$r=0$

$r=\frac{1}{2}$

La relación de recurrencia:

$$C_{m+1}=\frac{-C_m}{2m+2r+1}\qquad m=0,1,2,\dots$$

Encontrando las 2 soluciones:

Si $r=0$

$$C_{m+1}=\frac{-C_m}{2m+1}\qquad m=0,1,2,\dots$$

$m=0\qquad C_1=\frac{-C_0}{1}$

$m=1\qquad C_2=\frac{-C_1}{3}=\frac{C_0}{1*3}$

$m=2\qquad C_3=\frac{-C_2}{5}=\frac{-C_0}{1*3*5}$

$m=3\qquad C_4=\frac{-C_4}{7}=\frac{C_0}{1*3*5*7}$

$$C_n = \frac{(-1)^n C_0}{1 * 3 * 5 \dots \dots (2n-1)}$$

$$C_5 = \frac{-C_0}{1 * 3 * 5 * 7 * 9}$$

La solución:

$$y_1 = \sum_{n=1}^{\infty} \frac{(-1)^n C_0}{1 * 3 \dots (2n-1)} x^n + C_0$$

Si $r = \frac{1}{2}$

$$C_{m+1} = \frac{-C_m}{(2m+2r+1)} \qquad m = 0, 1, 2, \dots$$

$$= \frac{-C_m}{(2m+2)} = \frac{-C_m}{2(m+1)}$$

$m = 0 \quad C_1 = \frac{-C_0}{2}$

$m = 1 \quad C_2 = \frac{-C_1}{4} = \frac{C_0}{4 * 2} = \frac{C_0}{2^2 * 2!}$

$m = 2 \quad C_3 = \frac{-C_2}{6} = \frac{-C_0}{2^3 * 3!}$

$m = 3 \quad C_4 = \frac{-C_3}{8} = \frac{C_0}{2^4 * 4!}$

$$C_5 = \frac{-C_0}{2^5 * 5!}$$

$$\therefore C_n = \frac{(-1)^n C_0}{2^n * n!}$$

La solución:

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n C_0}{2^n * n!} x^{n+\frac{1}{2}}$$

Sean las funciones:

$$\left.\begin{aligned} y_1 &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \qquad x^n}{1 * 3 * 5 \dots (2n-1)} \\ y_2 &= \sum_{n=0}^{\infty} \frac{(-1)^n \qquad x^{n+\frac{1}{2}}}{2^n * n!} \end{aligned}\right\} son\ l.i.$$

y_1 converge $\forall x \ \ni \ |x| < +\infty$

$$x = -2 \qquad x^{n+\frac{1}{2}} = (-2)^{\frac{1}{2}} \qquad n = 0$$

y_2 converge $\forall x \qquad x \geq 0$

∴ La solución general es:

$$y = A_1 y_1 + A_2 y_2$$

Converge en $(0, +\infty)$